Example: Consider a billiard ball that has been hit "dead center".

\[ \sum F = -f_{cw}R = -Ia \]
\[ \sum F = -f_{cw} = ma_{cm} \]
\[ \sum F = N - w_{cm} = 0 \]

Initially, \( x(t_0) = 0 \), \( \omega(t_0) = 0 \), and \( v(t_0) = 0 \), so \( x(t) = v \). 

The "dead center" hit means \( \sum \tau = 0 \) so the ball will slide (not roll!) initially.

But static friction will slow the ball until \( v = R \omega \) and the ball will roll without slipping.

Knowing \( \omega \), then \( \Delta \tau = 0 \) so \( R \omega = 5 \mu g \).

Example continued

When does \( v = R \omega \)?
\[ v = \mu g \]
\[ \tau = 2\mu g \]

How far does the ball slide in this time?
\[ \Delta x = \frac{1}{2} v^2 + v \Delta t = \frac{12v^2}{49\mu g} \]

What is \( v_{cm} \) at this time?
\[ v_{cm} = \mu g \gamma \]

Example: What is the angular momentum of the Earth about its own rotation axis?
\[ L = I\omega = \frac{2}{5} MR^2 \omega \]
\[ = \frac{2}{5} MR^2 \left( \frac{2\pi}{T} \right) = 7.15 \times 10^{20} \text{ kg m}^2/\text{s} \]

Example: What is the angular momentum of the Earth revolving around the sun?
\[ L = r\omega = rMr_\omega \]
\[ = rM(\omega) = Mr^2 \omega = Mr^2 \left( \frac{2\pi}{T} \right) = 2.7 \times 10^{20} \text{ kg m}^2/\text{s} \]

Example: A skater is initially spinning at a rate of 10.0 rad/sec with \( I = 2.50 \text{ kg m}^2 \) when her arms are extended. What is her angular velocity after she pulls her arms in and reduces \( I \) to 1.60 \text{ kg m}^2?

On ice so we can ignore external torques.
\[ L = I\omega \]
\[ I\omega = I\omega \]
\[ \omega = \left( \frac{1}{I} \right) \omega = \frac{2.50 \text{ kg m}^2}{1.60 \text{ kg m}^2} \left( 0.0 \text{ rad/sec} \right) = 15.6 \text{ rad/sec} \]
Example: A child of mass 25.0 kg stands on the edge of a rotating platform of mass 150 kg and radius 4.00 m. The platform with the child on it rotates with an angular speed of 6.20 rad/sec. The child jumps off in the radial direction, what happens to the angular speed of that platform?

The moment of inertia for the platform with the child is

\[ I_p = I + m_r r^2 = \frac{1}{2} M r^2 + m_r r^2 \]

Assume there are no external torques so that \( L_i = L_f \)

\[ I_p \omega_i = I_p \omega_f \]

\[ \left( \frac{1}{2} M r^2 + m_r r^2 \right) \omega_i = \left( \frac{1}{2} M r^2 + m_r r^2 \right) \omega_f \]

Example continued

What happens to the platform if, a little while later, the child, starting from rest, jumps back on the platform?

From before:

\[ \omega_f = \frac{\omega_i}{1 + \frac{m_r}{M}} \]

\[ \omega_f = \frac{6.20 \text{ rad/sec}}{1 + \frac{25.0}{150}} = 4.65 \text{ rad/sec} \]

Example continued

What is the angular momentum of the child after the step off the platform?

\[ L_f = r m_r \omega_f = m_r (r \omega_f) = m_r r^2 \omega_f \]

Example continued

What is the angular momentum of the child after the step off the platform?

\[ L = r \times p \]

\[ L_i = r m v_i = \rho m (r \omega_i) = m_r r^2 \omega_i \]

From before:

\[ \omega_f = \frac{\omega_i}{1 + \frac{m_r}{M}} \]

Example continued

Equation (3) can be solved for \( T \):

\[ T = \frac{\frac{L}{2} + F_a (L)}{\sin \theta} = 352 \text{ N} \]

Equation (1) can be solved for \( F_a \):

\[ F_{a0} = T_a \cos \theta = 288 \text{ N} \]

Equation (2) can be solved for \( F_a \):

\[ F_{a0} = T_a \sin \theta = -2.00 \text{ N} \]
Example: Find the force exerted by the biceps muscle in holding a one liter milk carton with the forearm parallel to the floor. Assume that the hand is 35.0 cm from the elbow and that the upper arm is 30.0 cm long. The elbow is bent at a right angle and one tendon of the biceps is attached at a position 5.00 cm from the elbow and the other is attached 30.0 cm from the elbow. The weight of the forearm and empty hand is 18.0 N and the center of gravity is at a distance of 16.5 cm from the elbow.

\[ \sum F = F_c - F_{ea} - F_{ba} = 0 \]
\[ F_e = \frac{F_c + F_{ba}}{x_i} = 130 \text{ N} \]

Periodic Motion

Example: The displacement of a particle is given by the expression \( x(t) = (4.00 \text{ m}) \cos (3\pi t + \phi) \) where \( t \) is in seconds.

(a) What is the frequency and period of the motion?
\[
 f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{3\pi \text{ rad/sec}}{2\pi} = 1.5 \text{ Hz}
\]
\[
 T = \frac{1}{f} = 0.67 \text{ sec}
\]

(b) What is the amplitude of the motion?
\[
 A = 4.00 \text{ m}
\]

Example continued

(c) What is the displacement of the particle at \( t = 0.25 \) sec.
\[
 x(t) = (4.00 \text{ m}) \cos (3\pi t + \phi)
\]
\[
 x(0.25 \text{ sec}) = (4.00 \text{ m}) \cos (3\pi (0.25 \text{ sec}) + \phi) = 2.83 \text{ m}
\]

Example: A particle moving with simple harmonic motion travels a total distance of 20.0 cm in each cycle of its motion and its maximum acceleration is 50.0 m/s\(^2\).

(a) What is the angular frequency of the particle's motion?
\[
 x(t) = A \cos (\omega t + \phi), \quad \dot{x}(t) = -A \sin(\omega t + \phi), \quad \ddot{x}(t) = -A \omega^2 \cos(\omega t + \phi)
\]
\[
 a_{\text{max}} = A \omega^2
\]
\[
 \omega = \sqrt{\frac{a_{\text{max}}}{A}} = \sqrt{\frac{50 \text{ m/s}^2}{0.05 \text{ m}}} = 31.6 \text{ rad/s}
\]
Example continued

(b) What is the maximum speed of the particle?

\[ v_{\text{max}} = A\omega = 1.58 \text{ m/s} \]

Example continued

(c) What is the maximum acceleration of the mass?

\[ a_{\text{max}} = A\omega^2 = \left(\frac{\pi}{4}\right) \left(\frac{2}m\right) = 17.5 \text{ m/s}^2 \]

Example: A mass-spring system oscillates with amplitude 3.50 cm. The spring constant is 250 N/m and the mass is 0.500 kg.

(a) Determine the mechanical energy if the system.

\[ E(t) = U_{\text{kin}} = \frac{1}{2}kA^2 = 0.153 \text{ J} \]

(b) What is the maximum speed of the mass?

\[ E = K_{\text{max}} = \frac{1}{2}mv_{\text{max}}^2 \]

\[ v_{\text{max}} = \sqrt{\frac{2E}{m}} = 0.783 \text{ m/s} \]

Example continued

Example: A clock has a pendulum that performs one full swing every 1.0 sec. The object at the end of the string weighs 10.0 N. What is the length of the pendulum?

\[ T = 2\pi \sqrt{\frac{L}{g}} \]

Solving for \( L \):

\[ L = \frac{4T^2}{4\pi^2} \left(\frac{8\text{ m/s}^2}{10.0}\right)^2 = 0.25 \text{ m} \]

Example: Consider a rod of length \( L \) and mass \( m \) that is pivoted about one end. What is the period of the oscillations for this physical pendulum?

An FBD for the rod

Apply N2L in rotational form:

\[ \sum \tau = -mg \left(\frac{L}{2} \sin\theta\right) = I\alpha \]
Example continued

\[ \sum T = -mg \left( \frac{L}{2} \sin \theta \right) = 1a = \frac{1}{3} ML^2 \frac{d^2 \theta}{dt^2} \]
\[ \frac{d^2 \theta}{dt^2} = -\frac{3g}{2L} \sin \theta \]

Assume small amplitude oscillations so that \( \theta \ll 1 \) rad.

\[ \frac{d^2 \theta}{dt^2} = -\frac{3g}{2L} \sin \theta = -\frac{3g}{2L} \theta \]

Which now is the equation for SHM with \( \omega_0 = \frac{2g}{2L} \).

Gravitation

Example continued

Example: Three masses are arranged as shown. What is the force of mass 1 on mass 3?

\[ F_{13} = -\frac{GM_1 M_3}{r^2} \left( \frac{b}{\sqrt{a^2 + b^2}} \hat{x} + \frac{a}{\sqrt{a^2 + b^2}} \hat{y} \right) \]

Example: Three masses are arranged as shown. What is the net force on mass 2?

\[ F_{\text{net}} = F_{12} + F_{13} = \frac{GM_1 M_2}{r_2} \frac{GM_1 M_3}{r_3} \]

\[ = \frac{GM_1 M_2}{r_2} - \frac{GM_1 M_3}{r_3} \frac{a}{b} (\hat{x}) - \frac{GM_1 M_3}{-b} (\hat{x}) \]
Let $M_1$ = mass of the Earth.

$$F = \frac{GM_1 M_2}{r^2}$$

Here $F$ = the force the Earth exerts on mass $M_2$. This is the force known as weight, $w$.

$$w = \frac{GM_1 M_2}{r_e^2} = g M_2$$

Near the surface of the Earth, $g = 9.8 \text{ N/kg}$.

$$w = 9.8 \text{ N/kg} \times 9.8 \text{ m/s}^2$$

In general, $g = \frac{GM}{r^2}$

Note that $g = \frac{F}{m}$ is the gravitational force per unit mass. This is called the gravitational field strength. It is often referred to as the acceleration due to gravity.

Example: What is the weight of a 100 kg astronaut on the surface of the Earth (force of the Earth on the astronaut)? How about in low Earth orbit? This is an orbit about 300 km above the surface of the Earth.

On Earth: $w = mg = 980 \text{ N}$

In low Earth orbit: $w = mg(h) = m\left(\frac{GM}{(R_E + h)}\right) = 890 \text{ N}$

Their weight is reduced by about 10%. The astronaut is NOT weightless!

Escape Speed

With what speed does an object of mass $m$ need to be launched from the surface of a body with mass $M$ so that it can just make it to $r = \infty$?

$$E_i = E_f \quad K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_i^2 = \frac{GMm}{R} = 0 + 0$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

For the Earth $v_{esc} = 11.2 \text{ km/sec}$.

Example: What is the size of an object with the mass of the sun ($2 \times 10^{30} \text{ kg}$) and an escape velocity equal to the speed of light?

$$v_{esc} = \sqrt{\frac{2GM}{R}} = c$$

$$\frac{R}{c} = 3 \text{ km}$$

This size is called the Schwarzchild radius.
Consider $m_i$,

$$\sum F = \frac{Gm_i m_a}{a^2} = \frac{4\pi^2 m_i r_a}{T^2} = \frac{4\pi^2 m_i m_a}{a^2}$$

$$GT^2 (m_1 + m_2) = 4\pi^2 a^3$$

For planets in the solar system $m_1 + m_2 = M_{\text{sun}}$ and

$$GT^2 M_{\text{sun}} = 4\pi^2 a^3$$

Can use the Earth to rewrite Kepler’s third law as $T^2 = a^3$ if the period is in years and $a$ is in AUs.

$$1 \text{ AU} = 1.5 \times 10^8 \text{ km}$$

When Kepler formulated these laws they were empirical. It was not until much later when they were found to have a physical justification (Newtonian physics).

<table>
<thead>
<tr>
<th>Planet</th>
<th>$a$(AU)</th>
<th>$T$(years)</th>
<th>$a^3$</th>
<th>$T^2$</th>
<th>$a^3/T^2$</th>
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Is there a way to detect the presence of a planet short of direct imaging? Consider the figure on slide 10.

$$\sum F = \frac{Gm_i m_a}{(r_1 + r_2)^2} = \frac{v_1^2}{r_1} = m_1 m_a/r_1^2$$

$$\sum F_2 = \frac{Gm_i m_a}{(r_1 + r_2)^2} = \frac{v_2^2}{r_2} = m_2 m_a/r_2^2$$

Assume that $v_1 = v_2 = 0$

The center of mass is located at

$$r_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

The speed of the Sun in its orbit is

$$v_1 = \frac{2\pi a}{T} = 12.4 \text{ m/s}$$

The period is 11.86 years which is the Jupiter’s orbital period (assumed $v_1 = v_2 = 0$).

A distant astronomer viewing the Sun would be able to detect this motion of the Sun on the sky by noting the Doppler shift of features in the spectrum of the Sun. The astronomer would be able to deduce a estimates for the mass of Jupiter, its orbital period, eccentricity, and orbit radius.

Take $m_1$ to be the Sun and $m_2$ to be Jupiter, then $r_1 + r_2 = d$ is Jupiter’s distance from the Sun.

$$d = r_1 + r_2 = r + \left( \frac{m_1}{m_2} \right) = \left( 1 + \frac{m_1}{m_2} \right)$$

$$d = 778.4 \times 10^8 \text{ km}$$

$$\frac{m_1}{m_2} = 1050$$

Are measured values.

Use these values to determine that $r_1 = 7.40 \times 10^8 \text{ km}$. This is the size of the Sun’s orbit around the solar system’s barycenter. (This value is approximately the radius of the Sun.)
Rocket Motion

Accelerating systems where the mass of the accelerating body changes with
time. Momentum is still conserved.

Example: A rocket has an initial mass of \(7 \times 10^4\) kg and upon
firing its fuel burns at a rate of 250 kg/s. The exhaust velocity is 2500 m/s. If the rocket has a vertical ascent from
resting on Earth, how long after the engines fire will the rocket lift off?

Apply N2L:

\[
\sum F = F_{\text{thrust}} - W_w = ma
\]

What is the magnitude of the thrust and weight force on the rocket?

\[
F_{\text{thrust}} = \int u \, du = 6.25 \times 10^7\ N
\]

\[
W_w = mg = 6.86 \times 10^7\ N
\]

Example continued

Since the thrust is less than the weight force the rocket
cannot take off immediately.

When does the thrust equal the weight force?

\[
F_{\text{thrust}} = W_w
\]

\[
F_{\text{thrust}} = mg
\]

\[
m = \frac{F_{\text{thrust}}}{g} = 63800\ kg
\]

Example continued

The mass of the rocket is determined by \(m(t) = m_0 - \frac{dm}{dt}\).

When does \(m(t) = 63800\ kg\)?

\[
t = \frac{m(t) - m_0}{dm/dt} = 25\ sec
\]

Example continued

What function describes the fuel consumption?

\[
F_{\text{thrust}} = W_w
\]

\[
\frac{dm}{dt} = mg_u
\]

\[
\frac{u}{S_u} \int \frac{dm}{m} = \int dt
\]

\[
\frac{u}{S_u} (\ln m - \ln m_0) = t
\]

\[
m(t) = m_0 \exp \left( - \frac{S_u t}{u} \right)
\]
Example continued

Since only 20% of the mass can be spent, at what time is \( m(t) = 0.8m_0 \)?

\[
m(t) = m_0 \exp \left( -\frac{F_0 t}{m} \right) \\
0.8m_0 = m_0 \exp \left( -\frac{F_0 t}{m} \right) \\
t = \frac{m}{F_0} \ln 0.8 = 273 \text{ sec}
\]