Vector Addition

PHY 121 Finals Review
FSE Tutoring Centers
Spring 2016

Vector Addition: Place the vectors tip to tail. A vector may be moved any way you please provided that you do not change its length nor rotate it. The resultant points from the tail of the first vector to the tip of the second ($\mathbf{A}+\mathbf{B}$).

To add vectors together they must first be resolved into components. The $x$ ($y$) component of a vector is found by projecting the vector onto the $x$ ($y$) axis.

Example: Vector $\mathbf{A}$ has a length of 5.00 meters and points along the $x$-axis. Vector $\mathbf{B}$ has a length of 3.00 meters and points $120^\circ$ from the $+x$-axis. Compute $\mathbf{A}+\mathbf{B}=-\mathbf{C}$. 
Example continued

Example: At the instant a traffic light turns green, an automobile starts with a constant acceleration of 2.2 m/s². At the same instant a truck, traveling with a constant speed of 9.5 m/s, overtakes and passes the automobile.

(a) How much time will elapse before the automobile overtakes the truck?
(b) How fast will the car be traveling at that instant?
(c) Where do they meet?

What is the data you are given?
\begin{align*} 
\text{U(truck)} &= 9.5; \text{u(car)} = 0; \text{a(car)} = 2.2; \\
\text{The distance covered by both are the same; Hence in the same period of time...}
\end{align*}

\begin{align*}
\text{The components of C:} \quad &C_x = A_x + B_x = 5.00\text{m} + (-1.50\text{m}) = 3.50\text{m} \\
&\quad C_y = A_y + B_y = 0.00\text{m} + 2.60\text{m} = 2.60\text{m}
\end{align*}

\begin{align*}
\text{The length of C:} \quad &C = \sqrt{C_x^2 + C_y^2} \\
&= \sqrt{(3.50\text{m})^2 + (2.60\text{m})^2} \\
&= 4.36\text{m}
\end{align*}

\begin{align*}
\text{The direction of C:} \quad &\tan \theta = \frac{C_y}{C_x} \\
&= \frac{-1.50\text{m}}{3.50\text{m}} \\
&= 0.429 \\
&\theta = \tan^{-1}(0.429) = 23.2^\circ \text{ From the +x-axis}
\end{align*}
Example: A penny is dropped from the observation deck of the Empire State Building 369 m above the ground. With what velocity does it strike the ground? Ignore air resistance.

Example: You throw a ball into the air with speed 15.0 m/s; how high does the ball rise?

Example: An arrow is shot into the air with $\theta = 60^\circ$ (from the horizontal) and $v_0 = 20.0$ m/s. The arrow is released from a height of 1.80 m above the ground.

(a) What are $v_x$ and $v_y$ of the arrow when $t = 3$ sec?

(b) What are the $x$ and $y$ components of the displacement of the arrow during the 3.0 sec interval?
Example continued

The initial position of the arrow is

$$\mathbf{r}_i = 0\hat{x} + (1.8\text{ m})\hat{y}$$

The final position of the arrow is


dx

$$x_f = x_i + v_{ix}t = 30.0\text{ m}$$

The displacement is

$$\Delta \mathbf{r} = (30\text{ m})\hat{x} + (7.8\text{ m})\hat{y}$$

$$y_f = y_i + v_{iy}t - \frac{1}{2}gt^2 = 9.60\text{ m}$$

Determining the roots with quadratic formula

$$t = -0.10\text{ sec} + 3.64\text{ sec}$$

The distance traveled is:

$$\Delta x = v_{ix}t = 36.4\text{ m}$$

Example: How high does the arrow go?

The arrow rises until

$$\frac{dy}{dt} = 0$$

with

$$\frac{ds}{dt} = \frac{dy}{dt} \frac{dt}{d\theta}$$

The max height is reached when

$$t = \frac{v_{iy}}{g} = 1.77\text{ sec}$$

The y-coordinate of the arrow is

$$y(t = 1.77\text{ s}) = y_i + v_{iy}t - \frac{1}{2}gt^2 = 17.1\text{ m}$$

Relative motion

Example: You are traveling in a car (A) at 60 miles/hour east on a long straight road. The car (B) next to you is traveling at 65 miles/hour east. What is the speed of car B relative to car A?
Example continued:

\[ \Delta r_{BA} = \Delta r_{AG} + \Delta r_{BG} \]

Divide by \( \Delta t \):

\[ \frac{\Delta r_{BA}}{\Delta t} = \frac{\Delta r_{AG}}{\Delta t} + \frac{\Delta r_{BG}}{\Delta t} \]

From the picture:

\[ v_{BA} = v_{AG} - v_{BG} \]

\[ v_{BA} = 65 \text{ miles/hr east} - 60 \text{ miles/hr east} \]

\[ v_{BA} = 5 \text{ miles/hr east} \]

Example: You are traveling in a car (A) at 60 miles/hour east on a long straight road. The car (B) next to you is traveling at 65 miles/hour west. What is the speed of car B relative to car A?

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Example continued:

\[ \Delta r_{BA} = \Delta r_{AG} + \Delta r_{BG} \]

Divide by \( \Delta t \):

\[ \frac{\Delta r_{BA}}{\Delta t} = \frac{\Delta r_{AG}}{\Delta t} + \frac{\Delta r_{BG}}{\Delta t} \]

From the picture:

\[ v_{BA} = v_{AG} - v_{BG} \]

\[ v_{BA} = 65 \text{ miles/hr west} - 60 \text{ miles/hr east} \]

\[ v_{BA} = 125 \text{ miles/hr west} \]

Example: The current in a river has a steady speed of 0.5 m/s. A student swims upstream a distance of 1 km and then swims back to the starting point. If the student can swim 1.2 m/s in still water, how long does the round trip in the river take? How long would the same trip take in still water?

In still water:

\[ t = \frac{d}{v} = \frac{2000 \text{ m}}{1.2 \text{ m/s}} = 1670 \text{ s} \]

Or 835 seconds per leg
Upstream

\[ v_{up} = v_x - v_w = 0.7 \text{ m/s} \]

\[ t = \frac{d}{v} = \frac{1000 \text{ m}}{0.7 \text{ m/s}} = 1430 \text{ s} \]

Downstream

\[ v_{down} = v_x + v_w = 1.7 \text{ m/s} \]

\[ t = \frac{d}{v} = \frac{1000 \text{ m}}{1.7 \text{ m/s}} = 590 \text{ s} \]

The total time is 2020 seconds.

Example continued

\[ v_{new} = v_{jet} + v_{wind} \]

\[ = (300 \text{ mph}) \hat{x} + (100 \text{ mph}) \cos 30^\circ \hat{x} + (100 \text{ mph}) \sin 30^\circ \hat{y} \]

\[ = (387 \hat{x} + 50 \hat{y}) \text{ mph} \]

The magnitude and direction of the velocity are

\[ \sqrt{v_x^2 + v_y^2} = 390 \text{ mph} \]

\[ \tan \theta = \frac{v_y}{v_x} = 0.129 \]

The plane travels 390 mph 7.4° north of east.

Example: A jet moving initially with \( v = 300 \text{ mph} \) due east enters a region where the wind is blowing at 100 mph at 30° north of east. What is the new velocity of the jet?

Place the vectors tip-to-tail:

The plane travels 390 mph 7.4° north of east.

Free Body Diagrams

Use idealized models to account for all forces acting on each mass (body) involved in the system being analyzed.
Example: Find the tension in each cord of the system shown in the figure.

Example: A box slides across a rough surface. If the coefficient of kinetic friction is 0.3, what is the acceleration of the box?

FBD for box:

Apply Newton's 2nd Law:
\[ \sum F = N_b + f_{\text{kin}} + w_b = ma \]
\[ \sum F_x = -f_{\text{kin}} = ma \]
\[ \sum F_y = N_b - w_b = 0 \]

Example continued:

(1) \(- f_{\text{kin}} = ma \)
(2) \(N_b - w_b = 0 \rightarrow N_b = w_b = mg \)

From (1):
\(- f_{\text{kin}} = -\mu_b N_b = -\mu_b mg \rightarrow ma \)

Solving for \( a \):
\[ a = -\mu_b g \]
\[ a = -(0.3)(9.8 \text{ m/s}^2) = -2.94 \text{ m/s}^2 \]

Example: In the previous example, a box sliding across a rough surface was found to have an acceleration of \(-2.94 \text{ m/s}^2\). If the initial speed of the box is 10.0 m/s, how long does it take for the box to come to rest?

Know: \( a = -2.94 \text{ m/s}^2, v_0 = 10.0 \text{ m/s}, v_f = 0 \text{ m/s} \)
Want: \( \Delta t \)

\[ v_f = v_0 + at \]
\[ 0 = 10.0 \text{ m/s} + (-2.94 \text{ m/s}^2) \Delta t \]
\[ \Delta t = -\frac{10.0 \text{ m/s}}{-2.94 \text{ m/s}^2} = 3.40 \text{ sec} \]
Example: A 1.00 kg mass is at rest on a ramp that makes an angle of 20° with respect to the horizontal. For this situation \( \mu_s = 0.400 \). What is the magnitude of the static friction force?

\[ F_{\text{FBD for box}} \]

Apply Newton's 2\(^{nd}\) law

\[ \sum F = N - f_s - w = 0 \]

Example continued

The magnitude of the static friction force can be found from (1)

\[ f_s = \mu_s N \]

What is the magnitude of the maximum static friction force?

\[ f_{s,m} = \mu_k N \]

Example continued

If the angle of the ramp is changed to 40°, what is the acceleration of the mass? Take \( \mu_k = 0.35 \).

The FBD is unchanged, except \( f_{s} \) is now the kinetic friction force of the ramp on the box.

Apply Newton's 2\(^{nd}\) law

\[ \sum F = N - f_k - w = ma \]

Example continued

(1) \[ \sum F_x = -f_k + mg \sin \theta = 0 \]

(2) \[ \sum F_y = N - mg \cos \theta = 0 \]

The magnitude of the static friction force can be found from (1)

\[ f_s = \mu_s N \]

What is the magnitude of the maximum static friction force?

\[ f_{s,m} = \mu_k N = \mu_k mg \cos \theta = \mu_k mg \]

Example: A 3.00 kg mass rests on a frictionless tabletop. This mass is connected to a 5.00 kg mass by a light string as shown. Assume the pulley is massless.

a. Draw free body diagrams for the two masses and the pulley.
b. Apply Newton’s Second law to the two masses.

Dynamics of Circular Motion

\[
\begin{align*}
\mathbf{r}(t) &= r \cos \theta \hat{k} + r \sin \theta \hat{y} \\
\mathbf{v}(t) &= -r \omega \sin \theta \hat{k} + r \omega \cos \theta \hat{y} \\
\mathbf{a}(t) &= -r \omega^2 \cos \theta \hat{k} - r \omega^2 \sin \theta \hat{y}
\end{align*}
\]

Rotational motion can be related to translational motion by converting between Cartesian and polar coordinate systems.

The tangential velocity and radial acceleration of a body (constant speed case).

Example: The Hubble Space Telescope orbits the Earth at an altitude of about 600 km with an orbit period of about 100 minutes, what is Hubble’s orbital speed? (Assume a circular orbit.)

\[
\begin{align*}
\mathbf{v}_o &= \frac{\text{total distance}}{\text{total time}} = \frac{r \Delta \theta}{\Delta t} = \frac{2\pi}{T} = 7300 \text{m/s}
\end{align*}
\]

\[
\begin{align*}
r &= R_e + h = 6.98 \times 10^8 \text{ m} \\
T &= 6000 \text{ s}
\end{align*}
\]
Example continued

(b) What is HST's angular speed?

\[ \omega_m = \frac{\Delta \theta}{\Delta t} = \frac{2\pi}{T} = 1.05 \times 10^{-3} \text{ rad/sec} \]

Example: What is the magnitude of the radial acceleration of HST?

\[ a_r = \frac{v^2}{r} = \frac{7.63 \text{ m/s}^2}{0.78g} \]

Previously: Consider an object in uniform circular motion (speed-constant).

\[ r(t) = r \cos \theta \hat{x} + r \sin \theta \hat{y} \]

\[ v(t) = -r \cos \theta \hat{x} + r \sin \theta \hat{y} \]

\[ a(t) = -r \cos \theta \hat{x} - r \sin \theta \hat{y} \]

The magnitude of the (tangential) velocity is
\[ V = r \omega \]

The magnitude of the (radial) acceleration is
\[ a_r = \frac{v^2}{r} = \frac{V^2}{r} = \omega^2 r \]

Example: The rotor is an amusement park ride where people stand against the inside of a cylinder. Once the cylinder is spinning fast enough the floor drops out.

(a) What force keeps the people from falling out the bottom of the cylinder?

Draw an FBD for a person with their back to the wall:

It is the force of static friction.
Example continued
(b) If $\mu_s = 0.40$ and the cylinder has $r = 2.5$ m, what is the minimum angular speed of the cylinder so that the people don't fall out?

Apply Newton's 2nd law:
\[ \sum F = N_{cp} + f_{cap} + w_{ep} = ma \]

(1) $\sum F_c = N_{cp} = ma = m\omega^2r$

(2) $\sum F_i = f_{cap} - w_{ep} = 0$

From (2):
\[ f_{cap} = w_{ep} = mg \]
\[ \mu_s N_{cp} = \mu_s (mg) = m\omega^2r \]
\[ \omega = \sqrt{\frac{g}{\mu_s r}} = \sqrt{\frac{9.8 \text{ m/s}^2}{0.40 (2.5 \text{ m})}} \approx 3.13 \text{ rad/s} \]

Example continued
From (1):
\[ f_{cap} = m\omega^2r \]
\[ f_{cap} = \mu_s N_{cp} = \mu_s (mg) = m\omega^2r \]

Solving for $r$:
\[ r = \frac{2g}{\omega^2} \]

What is $\omega$?
\[ \omega = 33.3 \text{ rev/min} \]
\[ \omega = \frac{2\pi \text{ rad}}{60 \text{ sec}} \approx 3.5 \text{ rad/s} \]

\[ r = \frac{\mu_s g}{\omega^2} = \frac{(0.1)(9.8 \text{ m/s}^2)}{(3.50 \text{ rad/s})^2} \approx 0.08 \text{ m} \]

Example continued
Example: A coin is placed on a record that is rotating at 33.3 rpm. If $\mu_s = 0.1$, how far from the center of the record can the coin be placed without having it slip off?

Draw an FBD for the coin:

Apply Newton's 2nd law:
\[ \sum F = N_{cp} + f_{cap} + w_{ep} = ma \]

(1) $\sum F_c = N_{cp} = ma = m\omega^2r$

(2) $\sum F_i = f_{cap} - w_{ep} = 0$

Example: A coin is placed on a record that is rotating at 33.3 rpm. If $\mu_s = 0.1$, how far from the center of the record can the coin be placed without having it slip off?

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(2) $\sum F_i = f_{cap} - w_{ep} = 0$
Example continued

The apparent weight at the top of loop is

\[ N_{tv} = mg - m \left( \frac{v^2}{r} - g \right) \]

This is the minimum speed needed to make it around the loop.

\[ N_{tv} = 0 \text{ when } v = \sqrt{gr} \]

Example continued

Consider the car at the bottom of the loop, how does the apparent weight compare to the true weight?

**Example continued**

**Example continued**

Apply Newton's 2nd law:

\[ \sum F = N_{tv} - w_{cw} = ma \]

\[ \sum F = N_{tv} - w_{cw} = ma = m \frac{v^2}{r} \]

\[ N_{tv} = mg - \theta \frac{v^2}{r} \]

Here, \( N > mg \)

Example continued

From (2) the normal force is

\[ N_{tv} = mg \cos \theta \]

Using (1):

\[ -f_{cw} + mg \sin \theta = ma \]

\[ -\mu N_{tv} + mg \sin \theta = ma \]

\[ -\mu (mg \cos \theta) + mg \sin \theta = ma \]

\[ a = g(\sin \theta - \mu \cos \theta) = 3.67 \text{ m/s}^2 \]
Work and Energy

Example: The extinction of the dinosaurs and the majority of species on Earth in the Cretaceous Period (65 Myr ago) is thought to have been caused by an asteroid striking the Earth near the Yucatan Peninsula. The resulting ejecta caused widespread global climate change.

If the mass of the asteroid was $10^{16}$ kg (diameter in the range of 4-9 miles) and had a speed of 30.0 km/sec, what was the asteroid's kinetic energy?

$$K = \frac{1}{2}mv^2 = \frac{1}{2} \left(10^{16}\text{ kg}\right) \left(30\times10^3 \text{ m/s}\right)^2$$

$$= 4.5\times10^{24}\text{ J}$$

This is equivalent to ~10^9 Megatons of TNT.

Example: What is the net work done on a box of mass m that is being pushed along a rough surface as shown?

The forces and displacement (in unit vector notation) are

- $N_x = N_x \hat{x} = (mg + F \sin \theta) \hat{x}$
- $w_y = w_y (-\hat{y}) = -mg \hat{y}$
- $f_{fr} = f_{fr} (-\hat{x}) = -\mu_k (mg + F \sin \theta) \hat{x}$
- $F_x = F \cos \theta \hat{x} - F \sin \theta \hat{y}$
- $\Delta x = \Delta x \hat{x}$

The work done by the pushing force is:

$$W_{\Delta x} = F_x \cdot \Delta x = (F \cos \theta) \Delta x$$
The work done by the Normal force is:

\[ W_N = \mathbf{N} \cdot \Delta \mathbf{r} = 0 \]

The normal force is perpendicular to the displacement.

The work done by gravity is:

\[ W_g = \mathbf{w}_g \cdot \Delta \mathbf{r} = 0 \]

The force of gravity is perpendicular to the displacement.

The work done by kinetic friction is:

\[ W_f = \mathbf{F}_k \cdot \Delta \mathbf{r} = -\mu_k (mg \cos \theta) \Delta x \]

The net work done on the box is:

\[ W_{\text{net}} = W_g + W_f + W_{\text{ext}} + W_s = (F \cos \theta) \Delta x + 0 + 0 - \mu_k (mg \cos \theta) \Delta x = (F \cos \theta - \mu_k mg - \mu_k F \sin \theta) \Delta x \]

Example: A ball is tossed straight up. What is the work done by the force of gravity on the ball as it rises?

FRD for rising ball:

\[ W_g = \mathbf{w}_g \cdot \Delta \mathbf{r} = -mg \Delta y \]

\[ W < 0 \text{ and the KE of the ball decreases.} \]

Example: An ideal spring has \( k = 20.0 \text{ N/m} \). What is the amount of work done (by an external agent) to stretch the spring 0.40 m from its relaxed length?

\[ W = \int \mathbf{F} \cdot d\mathbf{r} = \int_0^{0.40 \text{ m}} k \Delta x = \frac{1}{2} \left( \frac{1}{2} \right) (20.0 \text{ N/m}) (0.40 \text{ m})^2 = 1.6 \text{ J} \]